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Thermal instability of viscoelastic fluids in porous media

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Abstract

A theoretical analysis of thermal instability driven by buoyancy forces is conducted in an initially quiescent, horizontal porous layer saturated by viscoelastic fluids. Modified Darcy's law is used to explain characteristics of fluid motion. The linear stability theory is employed to find the critical condition of the onset of convective motion. The results of the linear stability analysis show that the overstability is a preferred mode for a certain parameter range. Based on the results of linear stability analysis, a nonlinear stability analysis is conducted. The onset of convection has the form of a supercritical and stable bifurcation independent of the values of the elastic parameters. The Landau equations and the Nusselt number variations are derived for steady and oscillatory modes.

Keywords: Buoyancy-driven convection; Viscoelastic fluids; Porous media; Overstability; Nonlinear stability analysis

1. Introduction

The convective motion driven by buoyancy forces has attracted many researchers' interests. In this connection buoyancy-driven phenomena in porous media are actively under investigation. It is well known that the buoyancy-driven convection has a wide variety of engineering applications, such as geothermal reservoirs, agricultural product storage systems, packed-bed catalytic reactors, the pollutant transport in underground and the heat removal of nuclear power plants.

With Newtonian fluid system of slow heating Horton and Rogers [1] and Lapwood [2] conducted theoretical analysis on the critical condition of the onset of buoyancy-driven motion in fluid-saturated horizontal porous layers. Katto and Masuoka [3] showed experimentally the effect of Darcy number on the onset condition of buoyancy-driven convection. They employed Darcy's law to express the fluid characteristics in porous layers. In case of Newtonian fluid the stability analysis has been conducted under the principles of the exchange of stabilities. However, the viscoelastic fluid like polymeric liquids can exhibit markedly different stability properties. For the Rayleigh–Benard problem, Vest and Arpaci [4], and Koka and Ierley [5] analyzed overstability of Maxwell fluid and Oldroyd-B fluid, respectively. They confirmed that the buoyancy forces could induce the time-periodic instability before the exchange of stabilities.

The nonlinear stability analysis has close relation with the actual structure of the convection. Malkus and Veronis [6] conducted a nonlinear stability analysis of the Rayleigh–Benard problem by employing the power-series method. Since the nonlinear stability analysis is thought to be important to understand the structure of turbulence, it becomes one of the most active research fields. For the viscoelastic fluid Rosenbalt [7] performed a nonlinear stability analysis of the Rayleigh–Benard problem. He treated the bifurcation problem corresponding to the exchange of stabilities and overstability.

In the present study, linear and nonlinear stability analyses of initially quiescent, horizontal porous layers which are saturated with viscoelastic fluids are conducted. The effect of relaxation parameters on the variation of the Nusselt number with respect to Darcy–Rayleigh number is also investigated in the neighborhood of the critical conditions.

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Nomenclature

a	dimensionless wave number
d	fluid layer thickness
Da	Darcy number, K/d^2
$\mathbf{g}$	gravitational acceleration
K	permeability
Nu	Nusselt number, $qd/(k\Delta T)$
P	pressure
Ra	Rayleigh number, $g\beta\Delta T d^3/(\alpha\nu)$
Ra_{D}	Darcy–Rayleigh number, $Ra \times Da$
T	temperature
$\mathbf{u}$	velocity vector in Cartesian coordinates
w_1	dimensionless vertical velocity disturbance
(x, y, z)	dimensionless Cartesian coordinates

Greek symbols

α	effective thermal diffusivity
β	thermal expansion coefficient
ΔT	temperature difference
ε	strain retardation time
θ_1	dimensionless temperature disturbance, $g\beta d^3 T_1/(\alpha\nu)$
λ	stress relaxation time
μ	viscosity
ν	kinematic viscosity
ρ	density
σ	temporal growth rate
τ	dimensionless time
χ	perturbation parameter
ψ	stream function
ω	dimensionless oscillation frequency

Subscript

c	critical conditions

Superscripts

s	stationary mode
o	oscillatory mode

2. Linear stability analysis

2.1. Governing equations

The system considered here is an initially quiescent, fluid-saturated horizontal porous layer of depth "d" as shown in Fig. 1. The porous medium is homogeneous and isotropic, and saturated with viscoelastic fluid. The porous layer is heated slowly from below. Employing the Boussinesq approximation and the modified Darcy's model [8], for this system the governing equations of flow and temperature fields are expressed as

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

$$\frac{\mu}{K}\left(\bar{\varepsilon}\frac{\partial}{\partial t} + 1\right)\mathbf{u} = \left(\bar{\lambda}\frac{\partial}{\partial t} + 1\right)(-\nabla P + \rho\mathbf{g}) \tag{2}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)T = \alpha\nabla^2 T \tag{3}$$

$$\rho = \rho_{\mathrm{r}}[1 - \beta(T - T_{\mathrm{r}})] \tag{4}$$

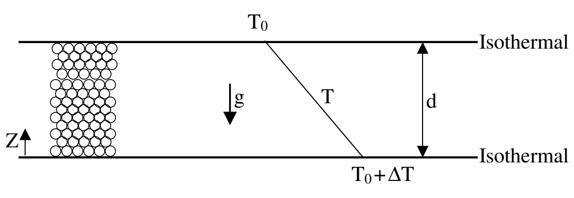

Fig. 1. Schematic diagram of system considered here.

where $\mathbf{u}$ is the velocity vector; K, the permeability; μ, the viscosity; $\bar{\varepsilon}$, the strain retardation time; $\bar{\lambda}$, the stress relaxation time; P, the pressure; T, the temperature; ρ, the density; $\mathbf{g}$, the gravitational acceleration; α, the effective thermal diffusivity and β is the thermal expansion coefficient. The subscript "r" represents the reference state.

The important parameters to describe the present system are

$$Da = \frac{K}{d^2}, \quad Ra = \frac{g\beta\Delta T d^3}{\alpha\nu}, \quad \varepsilon = \frac{\alpha\bar{\varepsilon}}{d^2} \quad \text{and} \quad \lambda = \frac{\alpha\bar{\lambda}}{d^2} \tag{5}$$

where ν denotes the kinematic viscosity and ΔT the temperature difference. For Newtonian fluid, the principle of exchange of stabilities holds and the critical condition is well represented by [1,2].

$$Ra_{\mathrm{D,c}} = Ra_{\mathrm{c}}Da = 4\pi^2 \tag{6}$$

2.2. Linear stability equations

Under the linear stability theory the disturbances caused by the onset of thermal convection can be formulated, in dimensionless form, in terms of the temperature disturbance θ_1, the vertical velocity disturbance w_1 and time τ by decomposing Eqs. (1)–(4).

$$\frac{1}{Da}\left(\varepsilon\frac{\partial}{\partial\tau} + 1\right)\nabla^2 w_1 = Ra\left(\lambda\frac{\partial}{\partial\tau} + 1\right)\nabla_1^2\theta_1 \tag{7}$$

$$\frac{\partial\theta_1}{\partial\tau} - w_1 = \nabla^2\theta_1 \tag{8}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ and $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The above equations are nondimensionalized by using d, d^2/α, α/d and ΔT as the length, time, velocity and the temperature unit, respectively. The proper boundary conditions are given by

$$w_1 = \theta_1 = 0 \quad \text{at } z = 0 \text{ and } z = 1 \tag{9}$$

The boundary conditions represent no flow through the boundaries and the fixed temperature on both boundaries.

According to the normal mode analysis, convective motion is assumed to exhibit the horizontal periodicity. Then the perturbed quantities can be expressed as follows:

$$[w_1(\tau,x,y,z),\theta_1(\tau,x,y,z)] = [w_1(z),\theta_1(z)]\exp[\mathrm{i}(a_x x + a_y y) + \sigma\tau] \tag{10}$$

where "i" is the imaginary number and σ is the temporal growth rate. With $\mathrm{Re}(\sigma) > 0$ the system will become unstable. For the Newtonian fluid, the principle of exchange of stabilities is satisfied, and therefore the imaginary part of σ is zero at the onset of motion. This means that the instability sets in as a steady secondary flow under the principle of exchange of stabilities. Substituting Eq. (10) into Eqs. (7) and (8) produces the usual amplitude functions in terms of the horizontal wave number $a = (a_x^2 + a_y^2)^{1/2}$.

$$\frac{1}{Da}(\varepsilon\sigma + 1)(\mathrm{D}^2 - a^2)w_1 = -Ra(\lambda\sigma + 1)a^2\theta_1 \tag{11}$$

$$\sigma\theta_1 - w_1 = (\mathrm{D}^2 - a^2)\theta_1 \tag{12}$$

where "D" is the z-directional differential operator, $\mathrm{D} = \mathrm{d}/\mathrm{d}z$. Eqs. (11) and (12) can readily be combined to yield

$$(\varepsilon\sigma + 1)(\mathrm{D}^2 - a^2)(\mathrm{D}^2 - a^2 - \sigma)\theta_1 = Ra_{\mathrm{D}}a^2(\lambda\sigma + 1)\theta_1 \tag{13}$$

And the boundary conditions, Eq. (9), are reduced to

$$\theta_1 = \mathrm{D}^2\theta_1 = 0 \quad \text{at } z = 0 \text{ and } z = 1 \tag{14}$$

2.3. Solution procedure and results

Examination of the boundary conditions, Eq. (14), and the stability equation, Eq. (13), shows that the required solution is

$$\theta_1 = A_n \sin n\pi z \quad (n = 1, 2, 3, \ldots) \tag{15}$$

where A_n is an arbitrary constant. Here, we are interested in the most dangerous mode, and our consideration is confined to the lowest-order mode, $n = 1$. The following characteristic equation for the most dangerous mode can be obtained by substituting Eq. (15) into Eq. (13).

$$(\varepsilon\sigma + 1)\pi^4 + (\varepsilon\sigma + 1)(2a^2 + \sigma)\pi^2 + (\varepsilon\sigma + 1)(a^4 + a^2\sigma) - Ra_{\mathrm{D}}a^2(\lambda\sigma + 1) = 0 \tag{16}$$

This equation can be rearranged as

$$\sigma^2\{\varepsilon\pi^2 + a^2\varepsilon\} + \sigma\{\varepsilon\pi^4 + \pi^2 + 2a^2\pi^2\varepsilon + a^2 + a^4\varepsilon - Ra_{\mathrm{D}}a^2\lambda\} + \{\pi^4 + 2a^2\pi^2 + a^4 - Ra_{\mathrm{D}}a^2\} = 0 \tag{17}$$

which may be expressed symbolically as

$$A\sigma^2 + B\sigma + C = 0 \tag{18}$$

From the elementary theory of algebraic equation, Eq. (18) may admit essentially two solutions, depending on whether the instability is steady or oscillatory. In steady (i.e. exchange of stabilities) case, we have $\sigma = 0$ at the critical condition. In this case the condition for the Darcy–Rayleigh number at which marginally stable steady mode exists can be obtained as

$$Ra_{\mathrm{D}} = \frac{\pi^4 + 2a^2\pi^2 + a^4}{a^2} \tag{19}$$

The critical wave number obtained by minimizing Ra_{D} with respect to a, i.e. satisfying $\partial Ra_{\mathrm{D}}/\partial a = 0$, is

$$a_{\mathrm{c}} = \pi \tag{20}$$

And, the corresponding critical Darcy–Rayleigh number for the steady case is

$$Ra_{\mathrm{D,c}}^{\mathrm{s}} = 4\pi^2 \tag{21}$$

The above results are independent of relaxation parameters and identical with those of Newtonian problem [1,2].

If $\mathrm{Im}(\sigma) \neq 0$ as $\mathrm{Re}(\sigma) \to 0$ for a disturbances, oscillatory instability which is sometimes called overstability sets in. For the case of oscillatory mode (i.e. overstability), it can be shown easily that a neutral overstability mode (i.e. $\sigma = \mathrm{i}\sigma_i$) occurs if

$$B = 0 \quad \text{and} \quad AC > 0 \tag{22}$$

From the first relation, the condition for the Darcy–Rayleigh number at which marginally stable oscillatory mode exists can be obtained as

$$Ra_{\mathrm{D}} = \frac{\varepsilon\pi^4 + \pi^2 + 2a^2\pi^2\varepsilon + a^2 + a^4\varepsilon}{a^2\lambda} \tag{23}$$

The critical wave number showing minimum Ra_{D} is

$$a_{\mathrm{c}}^2 = \sqrt{\pi^4 + \pi^2/\varepsilon} \tag{24}$$

It is interesting that the critical wave number is independent of $\bar{\lambda}$. And, the corresponding critical Darcy–Rayleigh number for the oscillatory case is

$$Ra_{\mathrm{D,c}}^{\mathrm{o}} = \frac{2\varepsilon\pi^4 + 2\pi^2 + 2\pi^2\varepsilon\sqrt{\pi^4 + \pi^2/\varepsilon} + \sqrt{\pi^4 + \pi^2/\varepsilon}}{\lambda\sqrt{\pi^4 + \pi^2/\varepsilon}} \tag{25}$$

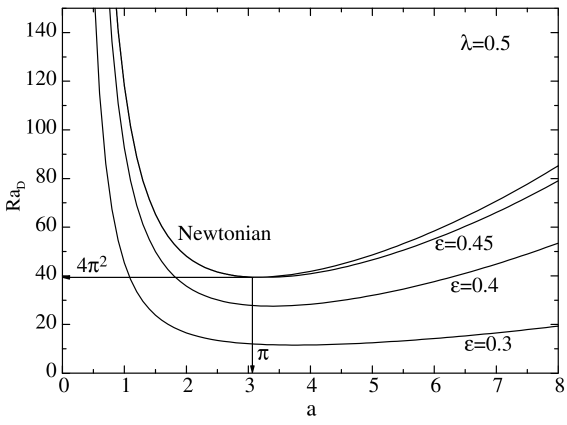

Fig. 2. Neutral stability curves for various ε for $\lambda = 0.5$.

From the second relation of Eq. (22) the overstability can occur for a particular wave number a only if the following inequality is satisfied.

$$(\lambda - \varepsilon) > \frac{1}{\pi^2 + a^2} \tag{26}$$

The dimensionless frequency of the neutral oscillatory mode is

$$\sigma_i^2 = \omega^2 = \frac{(\pi^2 + a^2)(\lambda - \varepsilon) - 1}{\varepsilon\lambda} \tag{27}$$

For $\lambda = 0.5$, the neutral stability curves are obtained as a function of ε as shown in Fig. 2. On each curve, the minimum for Ra_D will be the critical Darcy–Rayleigh number to mark the onset of convection. It is interesting that the neutral stability curve for $\varepsilon = 0.45$ is merged into the Newtonian case at the small wave number region. This means that Eq. (26) is an important criterion whether the convective motion shows overstability characteristics at a particular wave number. Eq. (26) reveals that overstability is likelier to occur as λ , which is related with the elasticity of fluid, increases and as ε, related with the viscous damping, decreases. The critical Darcy–Rayleigh number, $Ra_{\mathrm{D,c}}$ and the dimensionless frequency of neutral oscillatory mode, ω^2 in the ε–λ plane are given in Figs. 3 and 4. These figures show that the critical Darcy–Rayleigh numbers in the region of overstability is always smaller than those in the region of exchange of stabilities and the oscillation frequency decreases with increasing λ for a fixed ε.

3. Bifurcation of steady solutions

The linear stability theory gives us the critical Darcy–Rayleigh number, but does not predict the amplitude of convective motion. Here we use the perturbation method to find the bifurcation from the basic state at the

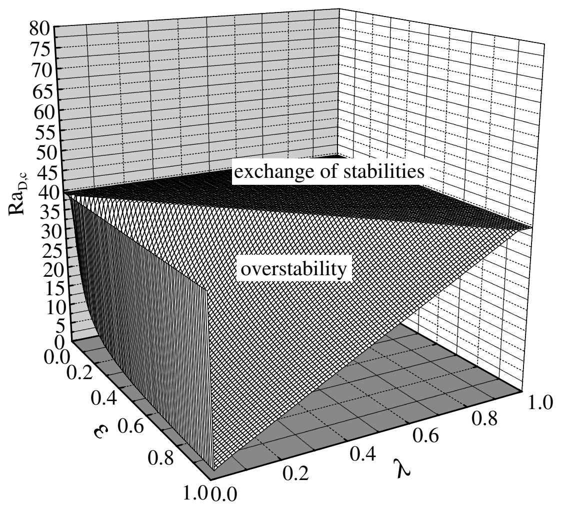

Fig. 3. The critical Darcy–Rayleigh number in the ε–λ space.

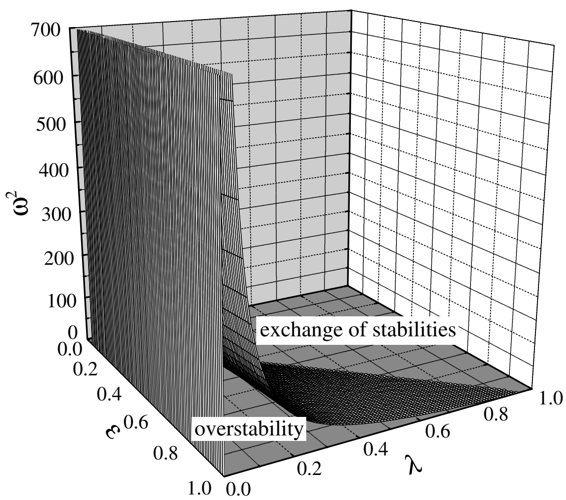

Fig. 4. The critical frequency in the ε–λ space.

value of $Ra_\mathrm{D} = Ra^\mathrm{s}_{\mathrm{D,c}}$. The results of this section have physical meaning only when the inequality of Eq. (26) does not hold. In the Rayleigh–Benard problem the convection cell takes the form of two-dimensional roll near the critical condition, so we simplify the three-dimensional problem to two-dimensional problem. Then, the velocity vector has only two components as follows:

$$\vec{u} = \vec{u}(x,z,t) = [u(x,z,t), 0, w(x,z,t)] \tag{28}$$

Using the continuity equation, the x- and z-direction velocities can be represented in terms of stream function $\psi(x,z,t)$ as

$$u = \frac{\partial\psi}{\partial z} \quad \text{and} \quad w = -\frac{\partial\psi}{\partial x} \tag{29}$$

The governing equations for two-dimensional flow and temperature fields including nonlinear terms may be expressed as follows:

$$\left(\varepsilon\frac{\partial}{\partial\tau}+1\right)\Delta^2\psi = -Ra_{\mathrm{D}}\left(\lambda\frac{\partial}{\partial\tau}+1\right)\frac{\partial\theta}{\partial x} \tag{30}$$

$$\frac{\partial\theta}{\partial\tau}+\frac{\partial\psi}{\partial x}-\Delta^2\theta = G \tag{31}$$

$$G = \frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial z}-\frac{\partial\psi}{\partial z}\frac{\partial\theta}{\partial x} \tag{32}$$

with the following boundary conditions

$$\psi = \theta = 0 \quad \text{at } z = 0 \text{ and } z = 1 \tag{33}$$

where $\Delta^2 = \partial^2/\partial z^2 + \partial^2/\partial x^2$.

Introducing a small perturbation parameter χ that indicates deviation from the critical state, the variables for a weak nonlinear state may be expanded as power series of χ [7].

$$\begin{aligned} Ra_{\mathrm{D}} &= Ra_{\mathrm{D,c}} + \chi^2 Ra_{\mathrm{D},2} + \cdots \\ \theta &= \chi\theta_1 + \chi^2\theta_2 + \chi^3\theta_3 + \cdots \\ \psi &= \chi\psi_1 + \chi^2\psi_2 + \chi^3\psi_3 + \cdots \end{aligned} \tag{34}$$

The scaling for the time variable τ is such that $\partial/\partial\tau = \chi^2\partial/\partial s$. $Ra_{\mathrm{D},1}$ in Eq. (34) is eliminated a priori, since it becomes zero due to the symmetry when the solvability condition is imposed. And, also $\theta_0\psi_0$ are eliminated, because the zeroth order solutions of temperature and velocity fields are $\theta_0 = 1 - z$ and $\psi_0 = 0$, which are considered in Eqs. (8) and (31), already. In the present case, $Ra_{\mathrm{D,c}} = Ra^{\mathrm{s}}_{\mathrm{D,c}}$ and $Ra_{\mathrm{D},2} > 0$ corresponds to the supercritical condition while $Ra_{\mathrm{D},2} < 0$ represent the subcritical condition.

In the first order problem, that is the linear stability problem whose solution is an eigen-function with undetermined amplitudes, $G_1 = 0$. The first order solution corresponding to $Ra_{\mathrm{D}} = Ra^{\mathrm{s}}_{\mathrm{D,c}}$ is

$$\theta_1 = A_1\cos ax\sin\pi z \quad \text{and} \quad \psi_1 = B_1\sin ax\sin\pi z \tag{35}$$

The undetermined amplitudes are related by

$$A_1 = -\frac{a}{c}B_1 \tag{36}$$

where $c = (\pi^2 + a^2)$.

In the second order problem $G_2 = (1/2)A_1B_1a\pi\times\sin 2\pi z$. The second order problem is reduced as follows:

$$\Delta^2\psi_2 = -Ra_{\mathrm{D}}\frac{\partial\psi_2}{\partial x} \tag{37}$$

$$\frac{\partial\psi_2}{\partial x} - \Delta^2\theta_2 = \frac{1}{2}A_1B_1a\pi\sin 2\pi z \tag{38}$$

The solution is

$$\theta_2 = \frac{a}{8\pi}A_1B_1\sin 2\pi z \quad \text{and} \quad \psi_2 = 0 \tag{39}$$

The nonlinear term of the third order problem is

$$G_3 = -\frac{a^2}{8}A_1B_1^2\cos ax\sin\pi z + \frac{a^2}{8}A_1B_1^2\cos ax\sin 3\pi z \tag{40}$$

And, the third order problem is reduced to

$$\bar{\varepsilon}\frac{\partial}{\partial s}\Delta^2\psi_1 + \Delta^2\psi_3 = -Ra_{\mathrm{D}}\bar{\lambda}\frac{\partial}{\partial s}\frac{\partial\theta_1}{\partial x} - Ra_{\mathrm{D}}\frac{\partial\theta_3}{\partial x} - Ra_{\mathrm{D},2}\frac{\partial\theta_1}{\partial x} \tag{41}$$

$$\begin{aligned}\frac{\partial\theta_1}{\partial s} + \frac{\partial\psi_3}{\partial x} - \Delta^2\theta_3 = &-\frac{a^2}{8}A_1B_1^2\cos ax\sin\pi z \\ &+\frac{a^2}{8}A_1B_1^2\cos ax\sin 3\pi z\end{aligned} \tag{42}$$

The solution has the following form

$$\begin{aligned}&\theta_3 = A_3\cos ax\sin\pi z + \cdots \quad \text{and} \\ &\psi_3 = B_3\sin ax\sin\pi z + \cdots\end{aligned} \tag{43}$$

The solvability condition that guarantees the existence of a solution for the third order equation requires the solution of the homogeneous part of the third order equation must be orthogonal to the inhomogeneous part of it. This yields the following Landau equation that describes the temporal variation of the amplitude A_1 of the convection cell:

$$\gamma\frac{\partial A_1}{\partial s} = \frac{a^2}{c}Ra_{\mathrm{D},2}A_1 - kA_1^3 \tag{44}$$

where $\gamma = (1 + \bar{\varepsilon}c - \bar{\lambda}a^2Ra^{\mathrm{s}}_{\mathrm{D,c}}/c)$ and $k = c^2/8$.

Steady state amplitude exists in the following form, when Ra_{D} is lager than $Ra^{\mathrm{s}}_{\mathrm{D,c}}$ (i.e. $Ra_{\mathrm{D},2} > 0$):

$$A_1^2 = \frac{8a^2}{c^3}Ra_{\mathrm{D},2} \tag{45}$$

The Nusselt number is represented by using Eq. (33) as follows:

$$Nu = 1 + \frac{2a^2}{c^2}(Ra_{\mathrm{D}} - Ra^{\mathrm{s}}_{\mathrm{D,c}}) \tag{46}$$

By substituting the critical condition into the above equation, the following relation can be obtained.

$$Nu = 1 + \frac{1}{2\pi^2}(Ra_{\mathrm{D}} - 4\pi^2) \tag{47}$$

This relation is identical to that of Newtonian case [9].

4. Bifurcation of periodic solutions

A slight modification of the methods applied in the previous section can be used to determine the bifurcation of the basic state at the value of $Ra_{\mathrm{D}} = Ra^{\mathrm{o}}_{\mathrm{D,c}}$. The results of this section have the physical meaning only when the inequality of Eq. (26) holds. As in the previous section, we simplify the three-dimensional problem to the two-dimensional one, and introduce the expressions of Eqs. (28) and (29) for the velocity field. Then, we can obtain the system of equations of Eqs. (30)–(33). In order to allow for the anticipated frequency shift along the bifurcation solution, we introduce the fast time scale of τ and the slow time scale of s. Therefore, the scaling of time variable is such that $\partial/\partial\tau = \partial/\partial\tau + \chi^2\partial/\partial s$.

In the first order problem $G_1 = 0$. Therefore, the first order problem reduces to the linear stability problem for overstability. The first order solution corresponding to $Ra_D = Ra^o_{D,c}$ is

$$\theta_1 = \{A_1(s)e^{i\omega\tau} + \overline{A}_1(s)e^{-i\omega\tau}\}\cos ax \sin \pi z \tag{48}$$

$$\psi_1 = \{B_1(s)e^{i\omega\tau} + \overline{B}_1(s)e^{-i\omega\tau}\}\sin ax \sin \pi z \tag{49}$$

where the overbar denotes complex conjugate, ω and a are taken to the critical values associated with $Ra_D = Ra^o_{D,c}$. The undetermined amplitudes are functions of the slow time scale, and are related by

$$B_1 = -\frac{c + i\omega}{a}A_1 \tag{50}$$

In the second order problem the nonlinear term G_2 is expressed as

$$G_2 = \frac{1}{2}\pi a\{A_1B_1e^{2i\omega\tau} + \overline{A}_1\overline{B}_1e^{-2i\omega\tau} + A_1\overline{B}_1 + \overline{A}_1B_1\} \times \sin 2\pi z \tag{51}$$

From the above relation, we can deduce that velocity and temperature fields have the terms having frequency 2ω and independent of the fast time scale. Thus we can express the second order temperature term as follows:

$$\theta_2 = \{\theta_{20} + \theta_{22}e^{2i\omega\tau} + \overline{\theta}_{22}e^{-2i\omega\tau}\}\sin 2\pi z \tag{52}$$

where θ_{22} and θ_{20} are temperature fields have the terms having frequency 2ω and independent of the fast time scale, respectively. The second order problem is reduced as

$$\left(\varepsilon\frac{\partial}{\partial s} + 1\right)\Delta^2\psi_2 = -Ra^o_{D,c}\left(\lambda\frac{\partial}{\partial s} + 1\right)\frac{\partial\theta_2}{\partial x} \tag{53}$$

$$\frac{\partial\theta_2}{\partial\tau} + \frac{\partial\psi_2}{\partial x} - \Delta^2\theta_2 = G_2 \tag{54}$$

And, the solutions of the second order problem are

$$\theta_{20} = \frac{a}{8\pi}\{A_1\overline{B}_1 + \overline{A}_1B_1\}, \quad \psi_{20} = 0 \tag{55}$$

and

$$\theta_{22} = \frac{\pi a A_1 B_1}{(8\pi^2 + 4i\omega)}, \quad \psi_{22} = 0 \tag{56}$$

In the third order problem the nonlinear term G_3 is expressed as:

$$G_3 = -\pi^2a^2\left\{\frac{(A_1\overline{B}_1 + \overline{A}_1B_1)B_1}{8\pi^2} + \frac{A_1B_1\overline{B}_1}{8\pi^2 + 4i\omega}\right\} \times \{\cos ax \sin \pi z - \cos ax \sin 3\pi z\} \tag{57}$$

Therefore, the third order problem has the solution of the following forms:

$$\theta_3 = A_3e^{i\omega\tau}\cos ax \sin \pi z + \cdots \tag{58}$$

$$\psi_3 = B_3e^{i\omega\tau}\sin ax \sin \pi z + \cdots \tag{59}$$

And, the third order problem is reduced as:

$$\left(\varepsilon\frac{\partial}{\partial\tau} + 1\right)\Delta^2\psi_3 + \varepsilon\frac{\partial}{\partial s}\Delta^2\psi_1 = -Ra^o_{D,c}\left(\lambda\frac{\partial}{\partial\tau} + 1\right)\frac{\partial\theta_3}{\partial x} - Ra^o_{D,c}\lambda\frac{\partial}{\partial s}\left(\frac{\partial\theta_1}{\partial x}\right) - Ra_{D,2}\frac{\partial\theta_1}{\partial x} \tag{60}$$

$$\frac{\partial\theta_1}{\partial s} + \frac{\partial\theta_3}{\partial\tau} + \frac{\partial\psi_3}{\partial x} - \Delta^2\theta_3 = -\pi^2a^2\left\{\frac{(A_1\overline{B}_1 + \overline{A}_1B_1)B_1}{8\pi^2} + \frac{A_1B_1\overline{B}_1}{8\pi^2 + 4i\varpi}\right\} \tag{61}$$

Under the stability conditions of Eqs. (24)–(27), these equations yield the following Landau equation that describes the temporal variation of the amplitude A_1 of the convection cell:

$$\gamma\frac{\partial A_1}{\partial s} = \frac{a^2}{(1 + i\omega\overline{\varepsilon})c}Ra_{D,2}A_1 - k|A_1|^2A_1 \tag{62a}$$

where

$$\gamma = \{1 + \overline{\varepsilon}(c + i\omega) - Ra^o_{D,c}\lambda a^2/c\}/(1 + i\omega\varepsilon) \tag{62b}$$

and

$$k = \pi^2\{(c^2 + \omega^2)/8\pi^2 + (c + i\omega)^2/8\pi^2 + (c^2 + \omega^2)/(8\pi^2 + 4i\omega)\} \tag{62c}$$

From the above, the following relations can be obtained

$$\frac{\partial|A_1|^2}{\partial s} = 2p_r|A_1|^2 - 2l_r|A_1|^4 \tag{63}$$

$$\frac{\partial(\mathrm{ph}(A_1))}{\partial s} = p_i - l_i|A_1|^2 \tag{64}$$

where $\{a^2/(1 + i\omega\overline{\varepsilon})c\}Ra_{D,2}\gamma^{-1} = p_r + ip_i$, $\gamma^{-1}k = l_r + il_i$ and $\mathrm{ph}(\cdot)$ represents the phase shift. The temporal evolution of $|A_1|$ can be expressed as a function of initial amplitude A_0 [10]:

$$|A_1|^2 = \frac{A_0^2}{(l_r/p_r)A_0^2 + [1 - (l_r/p_r)A_0^2]\exp(-2p_rs)} \tag{65}$$

For the case of $l_r > 0$ and $Ra_D > Ra_{D,c}$, i.e. $p_r > 0$, the above solution gives $|A_1| \sim A_0\exp(p_rs)$ as $s \to -\infty$ and $|A_1| \to 0$, just as in the linear theory, but

$$|A_1| \to \sqrt{\frac{p_r}{l_r}} \quad \text{as } s \to +\infty, \tag{66}$$

whatever the value of A_0. This is called supercritical stability, the base system being linearly unstable for $Ra_D > Ra_{D,c}$ but settling down as a new laminar flow. The steady state amplitude exists in the following form, when $Ra_{D,2}$ takes positive value

$$|A_1|^2 = \frac{p_r}{l_r} = \frac{8\sqrt{16\pi^4 + 16\omega^2}}{\sqrt{(24\pi^2c^2 + 8\pi^2\omega^2 - 8\omega^2 c)^2 + (16\pi^2\omega c + 8\omega c^2)^2}} \times Ra_{D,2} \tag{67}$$

The Nusselt number is represented by using Eq. (33) as

$$Nu = 1 - \chi^2 \frac{d\theta_2}{dz}\bigg|_{z=0} \tag{68}$$

By assembling Eqs. (52), (67) and (68) the area-averaged Nusselt number can be represented as

$$Nu = 1 - \chi^2 2\pi\{\theta_{20} + (\theta_{22} + \overline{\theta}_{22})\cos 2\omega\tau + (\theta_{22} - \overline{\theta}_{22})\mathrm{i}\sin 2\omega\tau\} \tag{69}$$

From the above equation the time- and area-averaged Nusselt number is expressed as follows:

$$Nu = 1 + \chi^2\left\{\frac{c}{2} + 2\pi^2\frac{\sqrt{c^2+\omega^2}}{\sqrt{64\pi^4 + 16\omega^2}}\right\}|A_1|^2 = 1 + \left\{\frac{c}{2} + 2\pi^2\frac{\sqrt{c^2+\omega^2}}{\sqrt{64\pi^4 + 16\omega^2}}\right\}\left\{\frac{Ra_D - Ra^{o}_{D,c}}{Ra_{D,2}}\right\}|A_1|^2 \tag{70}$$

The effects of elastic parameters on the heat transfer characteristics are summarized in Figs. 5 and 6. As shown in these figures, the heat transfer rate increases with increasing the relaxation parameter and decreasing the retardation parameter and the slope of Nu vs. Ra_D is nearly constant regardless of the relaxation parameter and the retardation parameters. In the exchange of stabilities regime, the heat transfer characteristics are identical with those of Newtonian case.

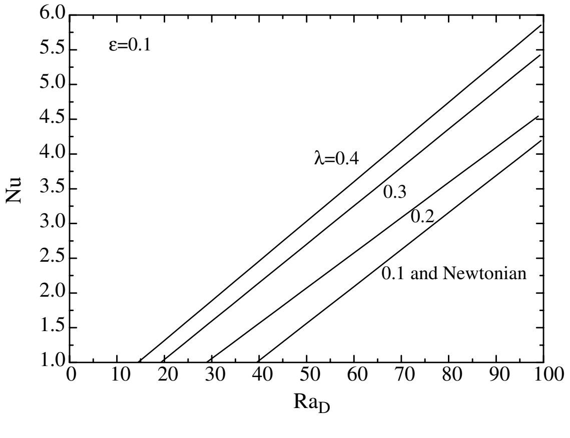

Fig. 5. The effect of relaxation parameter on the area- and time-averaged Nusselt number.

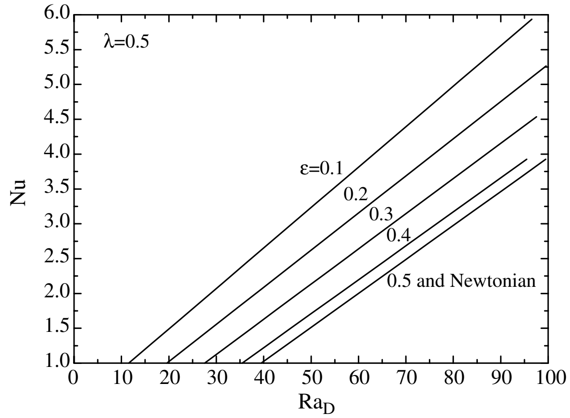

Fig. 6. The effect of retardation parameter on the area- and time-averaged Nusselt number.

5. Conclusions

The onset of buoyancy-driven motion in a horizontal porous layer saturated with viscoelastic fluid has been analyzed analytically by using linear and nonlinear stability theory. It is known that elasticity parameters are destabilizing factor and for a certain parameter range the overstability is a preferred mode. From the results of a bifurcation study, it can be known that the bifurcation of the present problem is supercritical and stable. The results of the present study, i.e. the critical Darcy–Rayleigh number, oscillation frequency and the heat transfer characteristics can be used to determine the elastic parameters of the non-Newtonian fluids in the porous media.

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